VEKTORANALYS

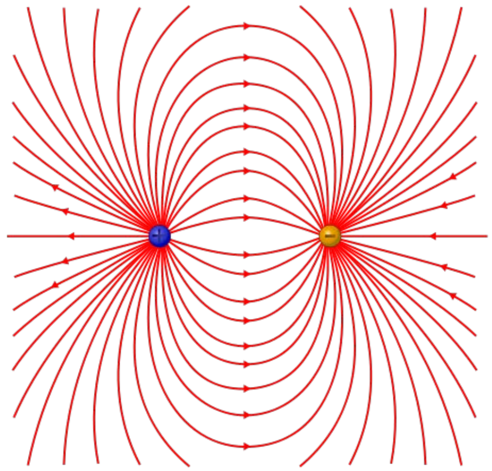
Kursvecka 3

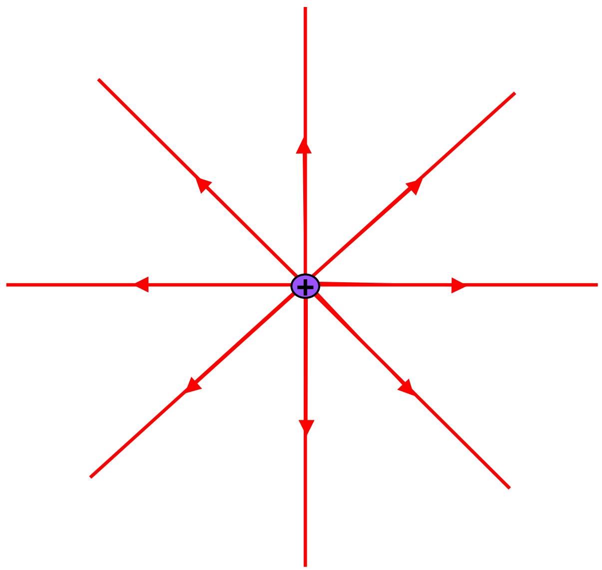
GAUSS's THEOREM
and
STOKES's THEOREM

Kapitel 6-7
Sidor 51-82

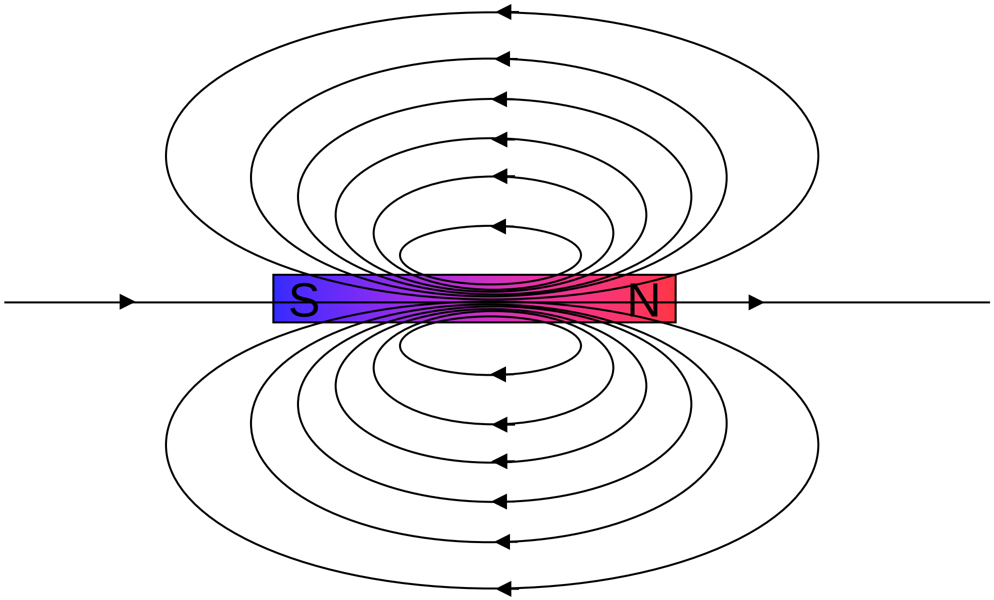
TARGET PROBLEM

ELECTRIC FIELD





MAGNETIC FIELD

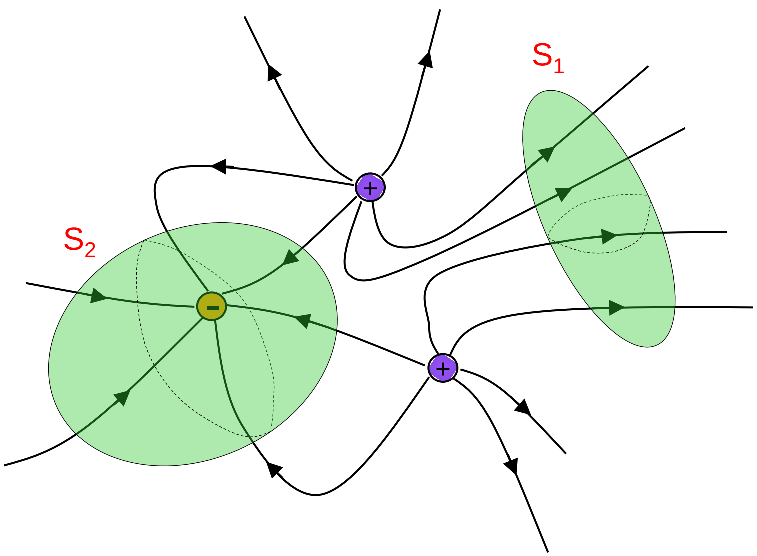


Magnetic monopoles do not exist in nature.

- **How can we express this information for $\bar{E}$ and $\bar{B}$ using the mathematical formalism?**

TARGET PROBLEM

Let's consider some ELECTRIC CHARGES and two closed surfaces, S_1 and S_2



S_1 does not contain any charge.
It has no sources and no sinks:
no field lines destroyed and
no field lines created inside S1

$$\iint_{S_1} \overline{E} \cdot d\overline{S} = 0$$

S_2 do contain a charge.
It has a sink. Field lines are destroyed inside S2

$$\iint_{S_2} \overline{E} \cdot d\overline{S} \neq 0$$

$$\boxed{\int_S \overline{E} \cdot d\overline{S} = \frac{Q}{\varepsilon_0} \quad \textbf{Guass's law}}$$

(see the 6th week of this course for details or "Teoretisk elektroteknik")

We want find the differential form of the Guass's law.
(i.e. to express the Guass's law without using integrals)

- to introduce the divergence of a vector field $\overline{A}$, $div\,\overline{A}$
- the Gauss's theorem $\iint_S \overline{A} \cdot d\overline{S} = \iiint_V div\overline{A} dV$

THE DIVERGENCE (DIVERGENSEN)

In cartesian coordinates, the divergence of a vector field $\bar{A}$ is:

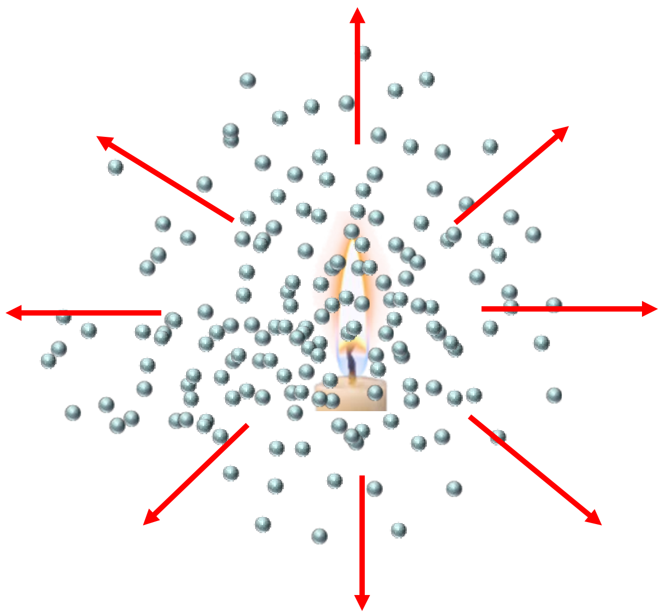
DEFINITION

$$\boxed{div\bar{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \qquad (1)$$

It is a measure of how much the field diverges (*or converges*) from (*to*) a point.

EXAMPLE:

- Assume that $\bar{A}$ is the velocity field of a gas.
- If heated, the gas will expand creating a velocity field that will diverge.
- The divergence of $\bar{A}$ in the heating point will be positive
- If cooled, the gas will contract creating a velocity field that will converge on the cooling position. The divergence will be negative
- The heating position is a **source** of the velocity field and the cooling position is a **sink** of the velocity



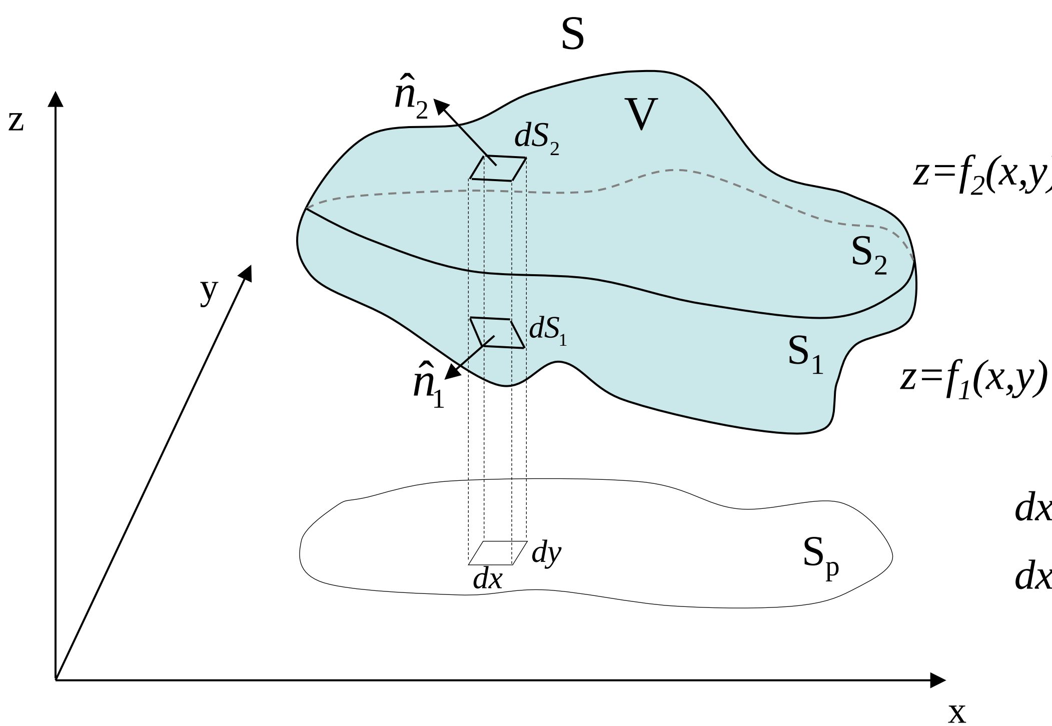
The divergence is a measurement of sources or sinks

(this will be more clear using the Guass's theorem)

THE GAUSS's THEOREM

$$\boxed{\iint_S \overline{A}\cdot d\overline{S} = \iiint_V div\overline{A}dV} \qquad (2)$$

where **S is a closed surface** that forms the boundary of the volume V and $\overline{A}$ is a continuously differentiable vector field defined on V.



$$dxdy = dS_2\hat{n}_2\cdot\hat{e}_z = d\overline{S}_2\cdot\hat{e}_z$$

$$dxdy = -dS_1\hat{n}_1\cdot\hat{e}_z = -d\overline{S}_1\cdot\hat{e}_z$$

THE GAUSS's THEOREM

PROOF

$$\iiint_V div\bar{A}dV = \iiint_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dxdydz =$$

$$\iiint_V \frac{\partial A_x}{\partial x} dxdydz + \iiint_V \frac{\partial A_y}{\partial y} dxdydz + \iiint_V \frac{\partial A_z}{\partial z} dxdydz$$

Let's calculate the last term:

$$\iiint_V \frac{\partial A_z}{\partial z} dxdydz = \iint_{S_p} dxdy \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial A_z}{\partial z} dz = \iint_{S_p} \left[A_z(x, y, f_2(x, y)) - A_z(x, y, f_1(x, y)) \right] dxdy =$$

$dxdy$ is the projection on S_p of the small element surfaces on dS_1 and dS_2.

Therefore: $dxdy = -\hat{e}_z \cdot \hat{n}_1 dS_1 = \hat{e}_z \cdot \hat{n}_2 dS_2$

$$= \iint_{S_2} A_z(x, y, f_2(x, y))\hat{e}_z \cdot \hat{n}_2 dS_2 + \iint_{S_1} A_z(x, y, f_1(x, y))\hat{e}_z \cdot \hat{n}_1 dS_1 = \iint_S A_z \hat{e}_z \cdot \hat{n} dS$$

Which means:

$$\iiint_V \frac{\partial A_z}{\partial z} dV = \iint_S A_z \hat{e}_z \cdot \hat{n} dS \qquad (3)$$

THE GAUSS's THEOREM

PROOF

In the same way we get:

$$\iiint\limits_V \frac{\partial A_x}{\partial x} dV = \iint\limits_S A_x \hat{e}_x \cdot \hat{n} dS \qquad (4)$$

$$\iiint\limits_V \frac{\partial A_y}{\partial y} dV = \iint\limits_S A_y \hat{e}_y \cdot \hat{n} dS \qquad (5)$$

Adding together equations (3), (4) and (5) we finally obtain:

$$\iiint\limits_V div\bar{A} dV = \iiint\limits_V \frac{\partial A_x}{\partial x} dxdydz + \iiint\limits_V \frac{\partial A_y}{\partial y} dxdydz + \iiint\limits_V \frac{\partial A_z}{\partial z} dxdydz =$$

$$\iint\limits_S A_x \hat{e}_x \cdot \hat{n} dS + \iint\limits_S A_y \hat{e}_y \cdot \hat{n} dS + \iint\limits_S A_z \hat{e}_z \cdot \hat{n} dS = \iint\limits_S \bar{A} \cdot d\bar{S}$$

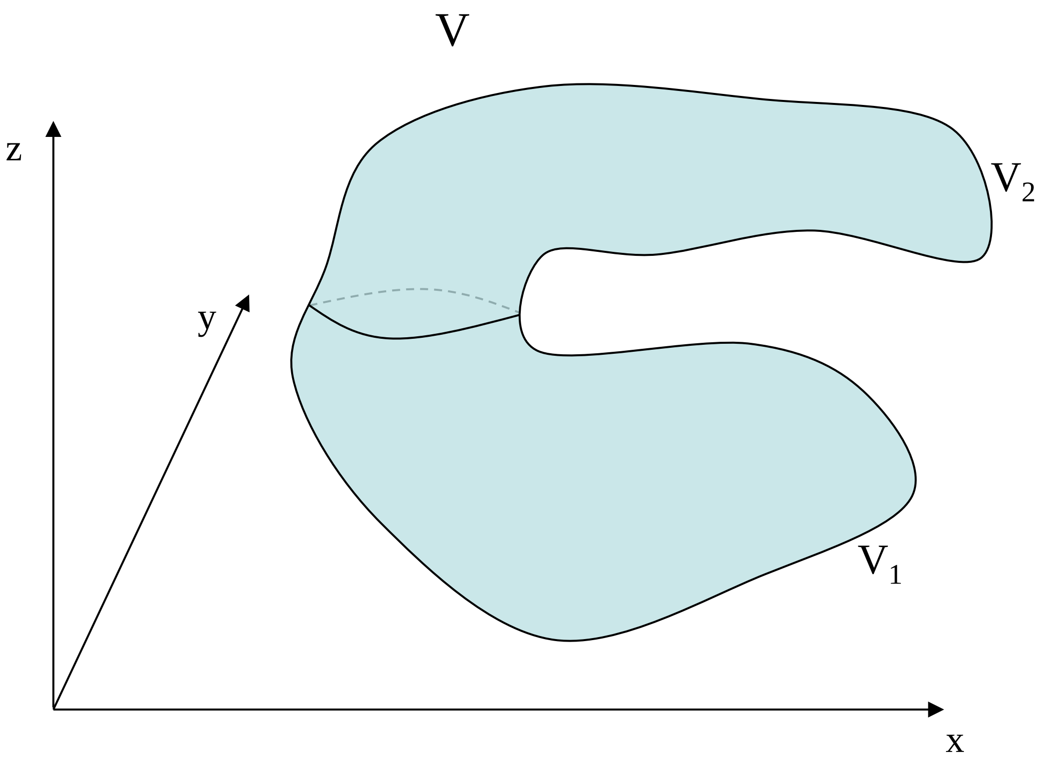
Rearrange in logic order the steps to prove the Gauss's theorem

- Add all the three terms together in order to obtain the flux of $\overline{A}$.
- Write down the volume integral of $div\overline{A}$
- Consider the projection of the surface element on the xy plane, it will be dxdy. The projection will identify a infinitesimal surface element (dS_2) on the lower surface.
  - Re-arrange the integrals in dS_1 and dS_2 in order to have obtain a flux integral of $(0,0,A_z)$.
- Consider a closed surface.
- Split the volume integral into three terms.
  - Consider only the term which depends on the z-derivative of A_z.
- Repeat the same for the terms which depend on the x-derivative of A_x and on the y-derivative of A_y.
  - Express dxdy in order to obtain dS_1 and dS_2.
  - Remove the z-derivative by solving the integral in dz. What will remain is just the integral in dxdy.
- Divide the surface in two parts, an upper surface and a lower surface and consider an infinitesimal surface element dS_1 on the upper surface.
- Write the expression that relates dxdy to dS_1 and dS_2.

THE GAUSS's THEOREM

PROOF

What if we consider a more complicated volume?



We divide the volume V in smaller and "simpler" volumes

$$V = V_1 + V_2 + \ldots = \sum_i V_i$$

$$\iiint_V divA dV = \sum_i \iiint_{V_i} divA dV =$$

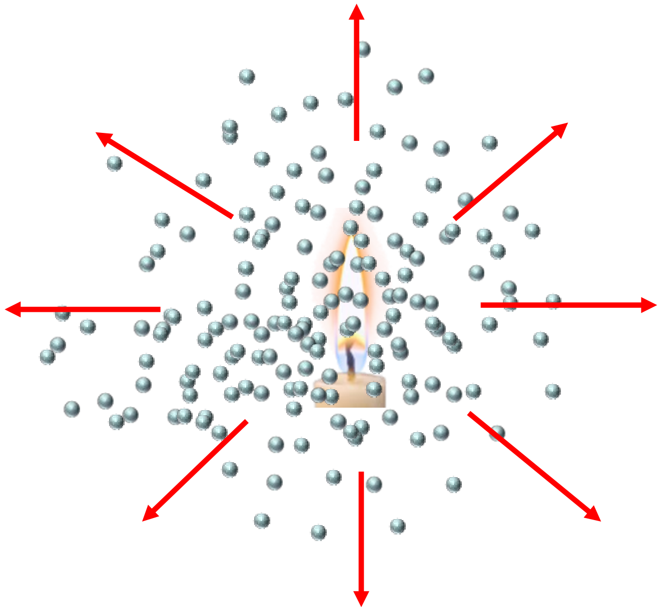
$$\sum_i \iint_{S_i} A \cdot dS = \iint_S A \cdot dS$$

PHYSICAL INTERPRETATION

Suppose that $\overline{v}(\overline{r})$ is the velocity field of a gas

Let's apply the Gauss' theorem to a volume V of the gas

$$\iint_S \overline{v} \cdot d\overline{S} = \iiint_V div(\overline{v}) dV$$



This term is the volume per second [m^3/s] that flows out *(in)* from the closed surface S

If there are no sinks and no sources,
then no gas flows in S
and no gas flows out from S.
This implies that the flow $\iint_S \overline{v} \cdot d\overline{S}$ is zero.
Therefore, $div(\overline{v}) = 0$

$div(\overline{v}) = 0 \Rightarrow$ No sink and no source

$div(\overline{v}) < 0 \Rightarrow$ flux is destroyed
and there is a sink

$div(\overline{v}) > 0 \Rightarrow$ flux is created
and there is a source

TARGET PROBLEM

Magnetic monopoles do not exist in nature.
What this implies, in terms of the magnetic field?

Magnetic monopoles do not exists $\Rightarrow$ the flux of **B** is zero

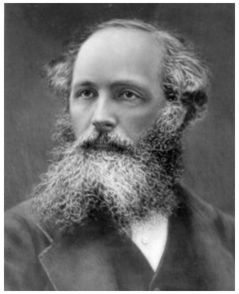
Let's apply the Gauss's theorem to the magnetic field:

Gauss $\Rightarrow$ $\iint_S \overline{B} \cdot d\overline{S} = \iiint_V div\overline{B} dV$

$$\left.\begin{array}{l} \iint_S \overline{B} \cdot d\overline{S} = \iiint_V div\overline{B} dV \\ \iint_S \overline{B} \cdot d\overline{S} = 0 \end{array}\right\} \rightarrow div\overline{B} = 0$$

Exercise: apply the Gauss's theorem to the Guass's law: $\iint \overline{E} \cdot d\overline{S} = \frac{Q}{\varepsilon_0}$

One of the four Maxwell's equations



WHICH STATEMENT IS WRONG?

1- The divergence of a vector field is a scalar (yellow)

2- The divergence is related to a measurement of the flux (red)

3- The Gauss' theorem translates a surface integral
into a volume integral (green)

4- The Gauss' theorem can be applied also to a non closed surface (blue)

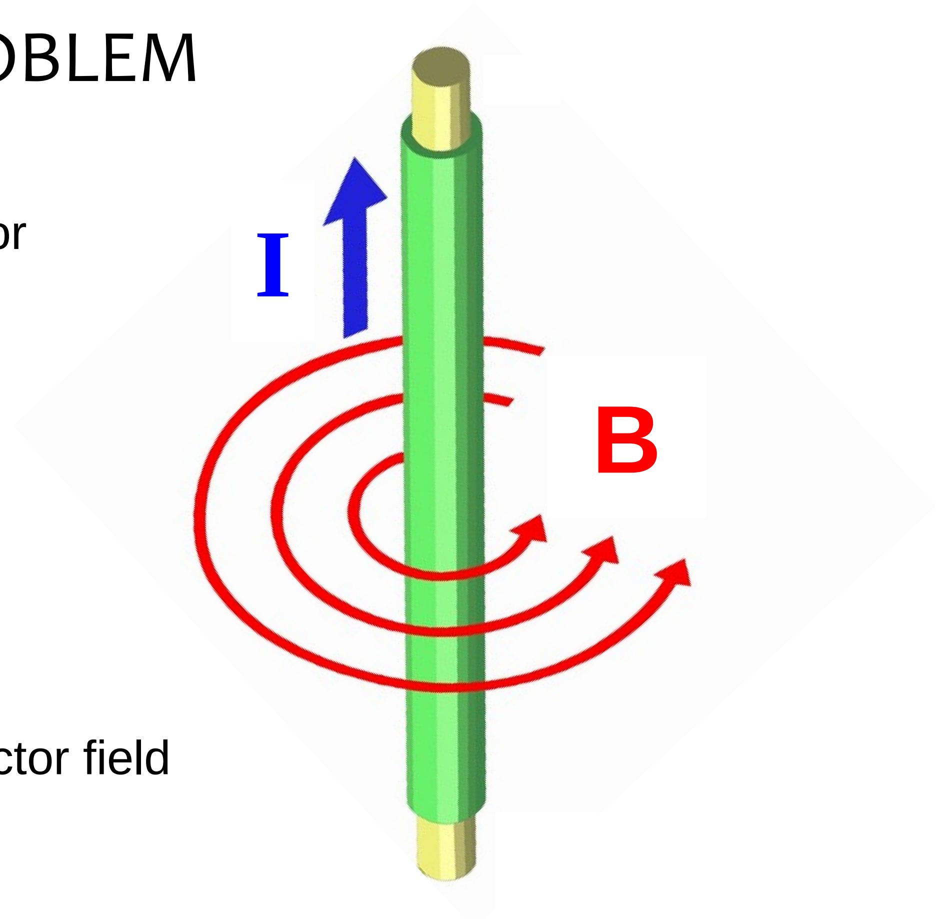
VEKTORANALYS

CURL (ROTATIONEN)

and

STOKES's THEOREM

TARGET PROBLEM



- The current $\overline{I}$ is flowing in a conductor
- How to calculate the magnetic field?

We need:

- Definition of the "curl" (or rotor) of a vector field

$$rot\,\overline{A}$$

- The Stokes' theorem

$$\oint_L \overline{A}\cdot dr = \iint_S rot\overline{A}\cdot d\overline{S}$$

- A law that relates the current with the magnetic field: the fourth Maxwell's equation (with static electric field): $rot\overline{B} = \mu_0\,\overline{j}$

(see "Teoretisk elektroteknik")

THE CURL (ROTATIONEN) $rot\,\overline{A}$

DEFINITION (in cartesian coordinate)

$$rot\,\overline{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

rot stands for "rotation"
In fact, the curl is a measure of how much the direction of a vector field changes in space, i.e. how much the field "rotates".

In every point of the space, $rot\,\overline{A}$ is a vector whose length and direction characterize the rotation of the field $\overline{A}$.

The direction is the axis of rotation of $\overline{A}$

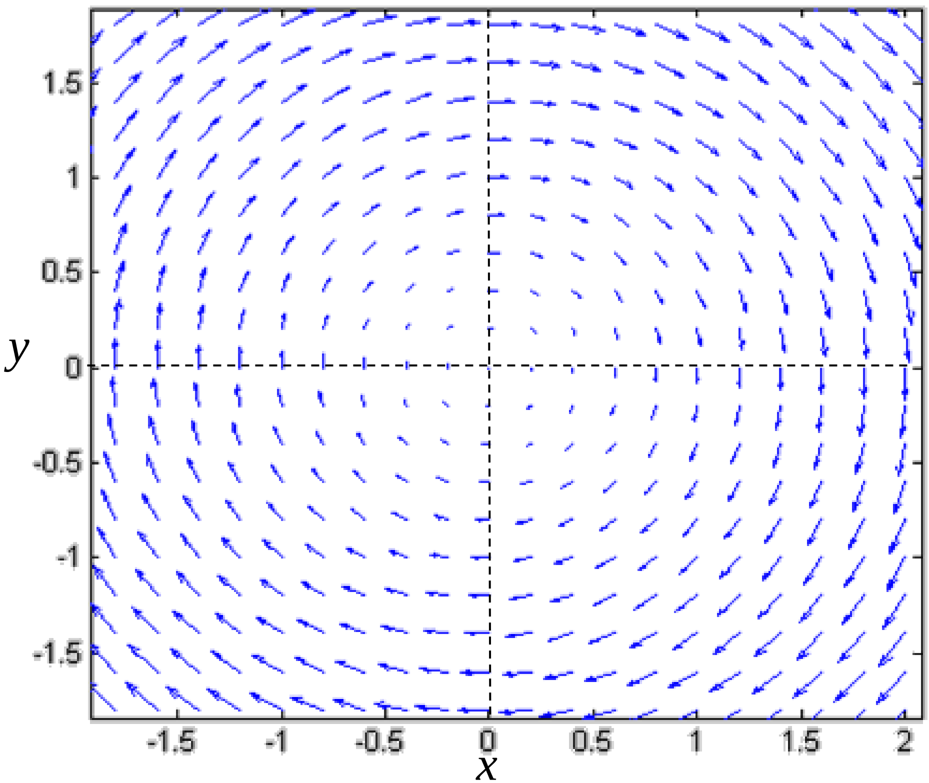
The magnitude is the magnitude of rotation of $\overline{A}$

THE CURL $rot\bar{A}$

EXAMPLE

$$\bar{A}(x, y, z) = (y, -x, 0)$$

Exercise: calculate the curl of $\bar{A}$

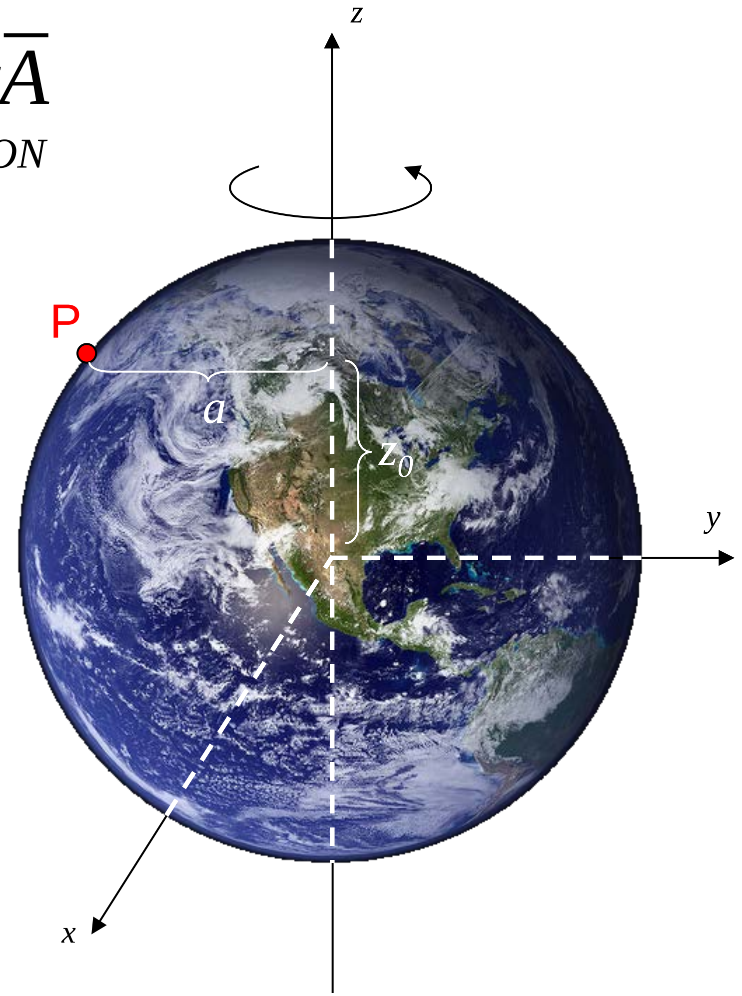


Direction: the direction is the axis of rotation, i.e. perpendicular to the plane of the figure
The sign (negative, in this case) is determined by the right-hand rule

Magnitude: the amount of rotation
In this example, it is constant and independent of the position, i.e. the amount of rotation is the same at any point.

THE CURL $rot\bar{A}$

PHYSICAL INTERPRETATION



Consider the rotation of a rigid body around the z-axis

The coordinates of a point P on the body located at the distance a from the z-*axis* and at $z=z_0$ changes in time:

$$x(t) = a\cos\omega t$$

$$y(t) = a\sin\omega t$$

$$z = z_0$$

The velocity of the point P is:

$$\left.\begin{aligned} v_x(t) &= -a\omega\sin\omega t = -\omega y(t) \\ v_y(t) &= a\omega\cos\omega t = \omega x(t) \\ v_z &= 0 \end{aligned}\right\} \Rightarrow \bar{v} = (-\omega y, \omega x, 0)$$

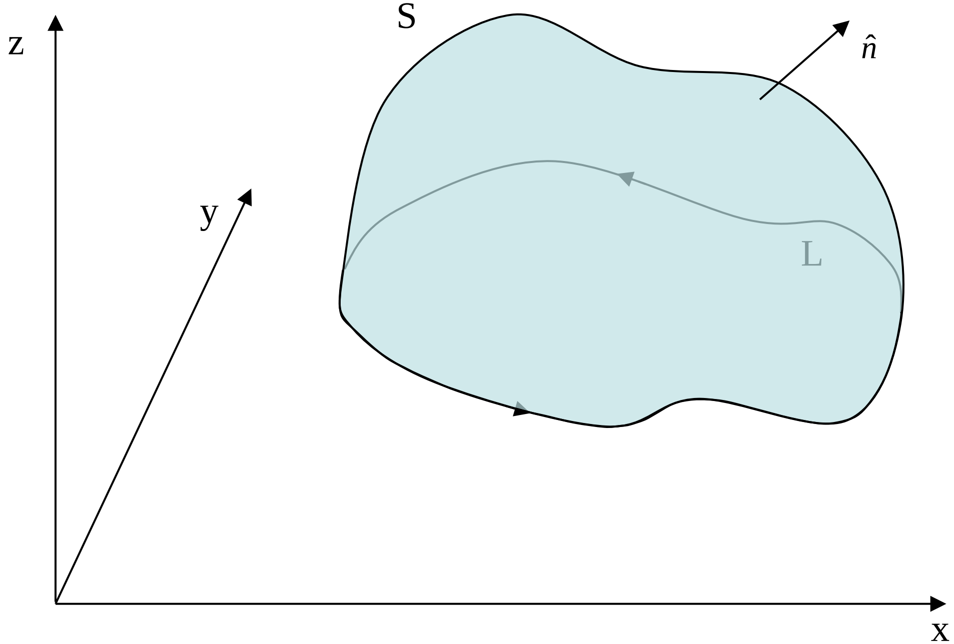
Therefore $rot\,\bar{v} = (0, 0, 2\omega)$

$$\bar{\omega} = \frac{1}{2} rot\,\bar{v}$$

THE STOKES' THEOREM

$$\oint_L \overline{A} \cdot d\overline{r} = \iint_S rot\overline{A} \cdot d\overline{S}$$

where $\overline{A}$ is a vector field, **<u>*L* is a closed curve</u>** and S is a surface whose boundary is defined by L.
$\overline{A}$ must be continuously differentiable on S



THE STOKES' THEOREM

PROOF

Five steps:

1. We divide S in "many" "smaller" *(infinitesimal)* surfaces:

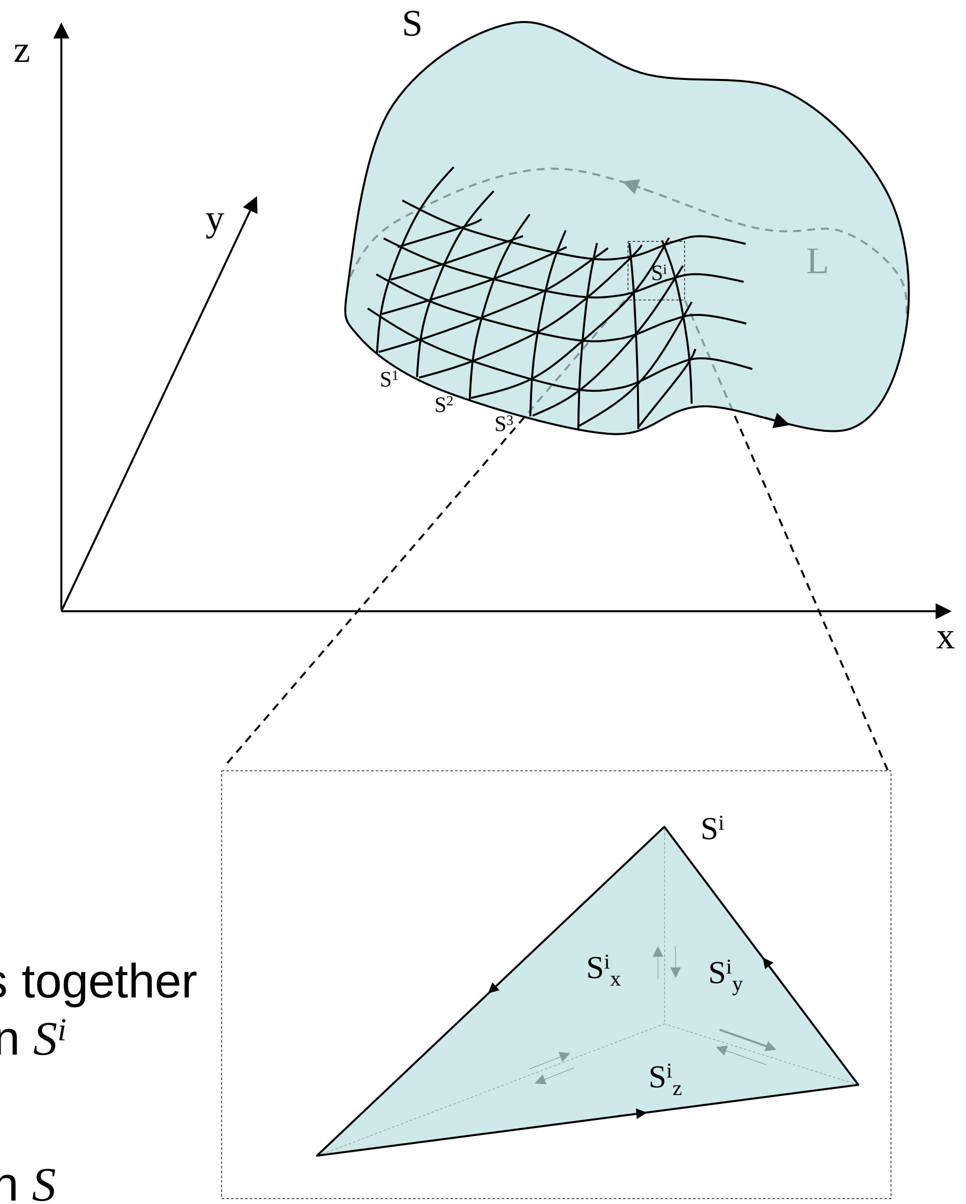
$$S = \sum_i S^i$$

2. We project S^i on:
   the xy-plane S^i_z
   the yz-plane S^i_x
   the xz-plane S^i_y

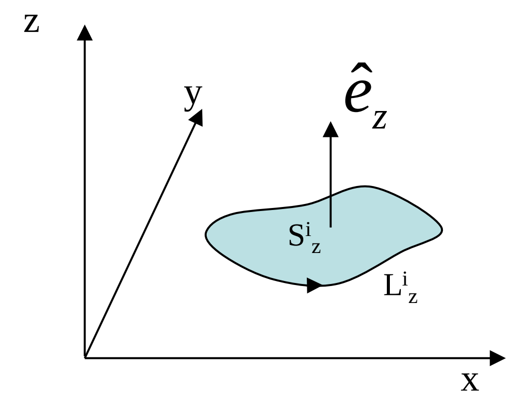
3. We prove the Stokes' theorem on S^i_z,
   (the only difficult part)

4. We add the results for the projections together and we obtain the Stokes' theorem on S^i

5. We add the results for S^i together and we obtain the Stokes' theorem on S



THE STOKES' THEOREM



PROOF
Let's consider the plane surface S^i_z
located in the xy-plane (i.e. z=constant=z_0)
with boundary defined by the curve L^i_z

Let's calculate $\oint_{L^i_z} \bar{A} \cdot d\bar{r}$

$$\oint_{L^i_z} \bar{A} \cdot d\bar{r} = \oint_{L^i_z} \underbrace{A_x(x,y,z_0)dx}_{\text{Term 1}} + \underbrace{A_y(x,y,z_0)dy}_{\text{Term 2}} + \underbrace{A_z(x,y,z_0)dz}_{\text{Term 3}}$$

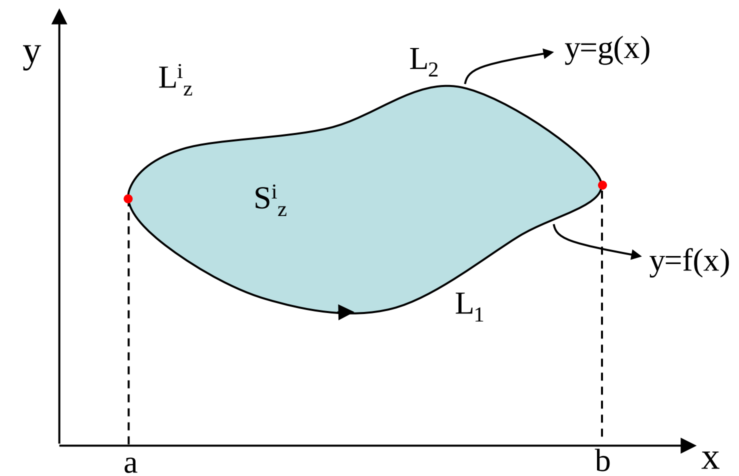
Term 3 =0 *(z=constant!$\Rightarrow$ dz=0)*

Term 1

$$\oint_{L^i_z} A_x(x,y,z_0)dx = \oint_{L_1+L_2} A_x(x,y,z_0)dx =$$

$$\int_{L_1} A_x(x,y,z_0)dx + \int_{L_2} A_x(x,y,z_0)dx =$$

$$\int_a^b A_x(x, f(x), z_0)dx + \int_b^a A_x(x, g(x), z_0)dx =$$



THE STOKES' THEOREM

PROOF

$$= \int_a^b A_x(x, f(x), z_0)dx - \int_a^b A_x(x, g(x), z_0)dx = \int_a^b \left[A_x(x, f(x), z_0) - A_x(x, g(x), z_0)\right]dx =$$

$$\int_a^b \int_{g(x)}^{f(x)} \frac{\partial A_x(x, y, z_0)}{\partial y} dxdy = -\int_a^b \int_{f(x)}^{g(x)} \frac{\partial A_x}{\partial y} dxdy = -\iint_{S^i{}_z} \frac{\partial A_x}{\partial y} dxdy$$

Therefore we get:

Term 1 $$\oint_{L^i{}_z} A_x(x, y, z_0)dx = -\iint_{S^i{}_z} \frac{\partial A_x}{\partial y} dxdy$$

In a similar way:

Term 2 $$\oint_{L^i{}_z} A_y(x, y, z_0)dx = \iint_{S^i{}_z} \frac{\partial A_y}{\partial x} dxdy$$

It is the z-component of $rot\bar{A}$!!

Adding Term 1, Term 2 and Term 3:

$$\oint_{L^i{}_z} \bar{A} \cdot d\bar{r} = \iint_{S^i{}_z} \left(\overbrace{\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}} \right) dxdy$$

THE STOKES' THEOREM

So can rewrite it as:

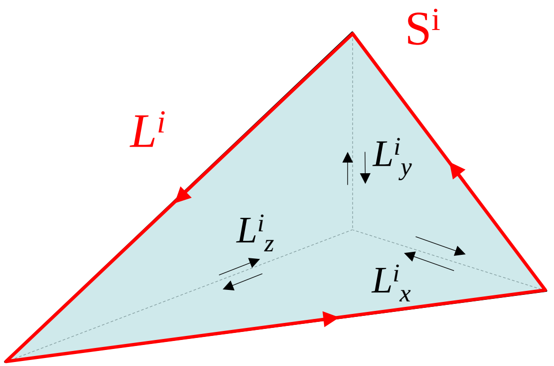
$$\oint_{L^i{}_z} \overline{A}\cdot d\overline{r} = \iint_{S^i{}_z} (rot\overline{A})_z\, dxdy = \iint_{S^i} (rot\overline{A})_z\, \hat{e}_z \cdot d\overline{S}$$

$$dxdy = \hat{e}_z \cdot \hat{n} dS = \hat{e}_z \cdot d\overline{S}$$

In a similar way we have:

$$\oint_{L^i{}_y} \overline{A}\cdot d\overline{r} = \iint_{S^i} (rot\overline{A})_y\, \hat{e}_y \cdot d\overline{S}$$

$$\oint_{L^i{}_x} \overline{A}\cdot d\overline{r} = \iint_{S^i} (rot\overline{A})_x\, \hat{e}_x \cdot d\overline{S}$$



Now let's add everything together:

$$\oint_{L^i{}_x} \overline{A}\cdot d\overline{r} + \oint_{L^i{}_y} \overline{A}\cdot d\overline{r} + \oint_{L^i{}_z} \overline{A}\cdot d\overline{r} = \oint_{L^i} \overline{A}\cdot d\overline{r}$$

$$\iint_{S^i} (rot\overline{A})_x\, \hat{e}_x \cdot d\overline{S} + \iint_{S^i} (rot\overline{A})_y\, \hat{e}_y \cdot d\overline{S} + \iint_{S^i} (rot\overline{A})_z\, \hat{e}_z \cdot d\overline{S} = \iint_{S^i} rot\overline{A}\cdot d\overline{S}$$

Rearrange in logic order the steps to prove the Stokes' theorem

- Consider only the integral in dx and prove that $\int_{L^i_z} A_x(x, y, z_0)dx = -\iint_{S^i_z} \frac{\partial A_x}{\partial y} dxdy$

- add together the expressions for the integrals in S^i_x to S^i_y and S^i_z obtaining: $\int_{L^i} \overline{A} \cdot d\overline{r} = \iint_{S^i} rot\overline{A} \cdot d\overline{S}$

- Prove the Stokes' theorem on S^i_z:

- Write the line integral of the vector field along the boundary of S^i_z and split the integral into three terms.

- Consider a closed path and a surface whose boundary is defined by the closed path.

-Divide the surface in small areas S^i and consider the projection of S^i on the xy, yz, xz planes

-Repeat the same for the integral in dy and dz

-Prove the Stokes' theorem on S: add together all the expressions obtained for S^i

-Rewrite $dxdy$ to obtain $\int_{L^i_z} \overline{A} \cdot d\overline{r} = \iint_{S^i} (rot\overline{A})_z \hat{e}_z \cdot d\overline{S}$

-Prove the Stokes' theorem on S^i:

-Add the three integrals in dx, dy and dz to obtain $\int_{L^i_z} \overline{A} \cdot d\overline{r} = \iint_{S^i_z} (rot\overline{A})_z dxdy$

-Repeat the same procedure for S^i_x and S^i_y

THE STOKES' THEOREM

PROOF

$$\oint_{L^i} \overline{A} \cdot d\overline{r} = \iint_{S^i} rot\overline{A} \cdot d\overline{S}$$

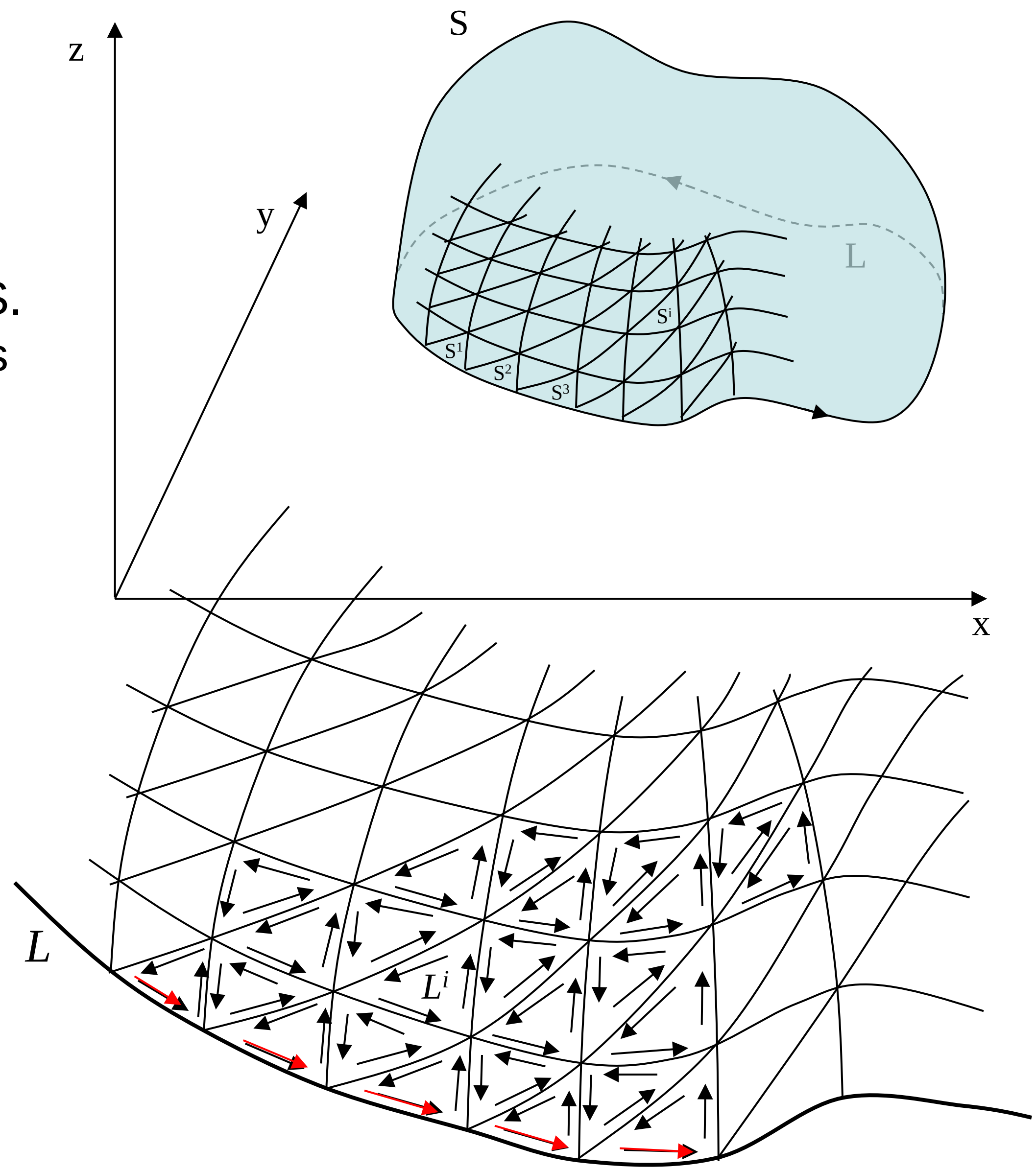
But we are interested in the whole S.
So we add these small contributions altogether:

$$\underbrace{\sum_i \iint_{S^i} rot\overline{A} \cdot d\overline{S}} = \iint_S rot\overline{A} \cdot d\overline{S}$$

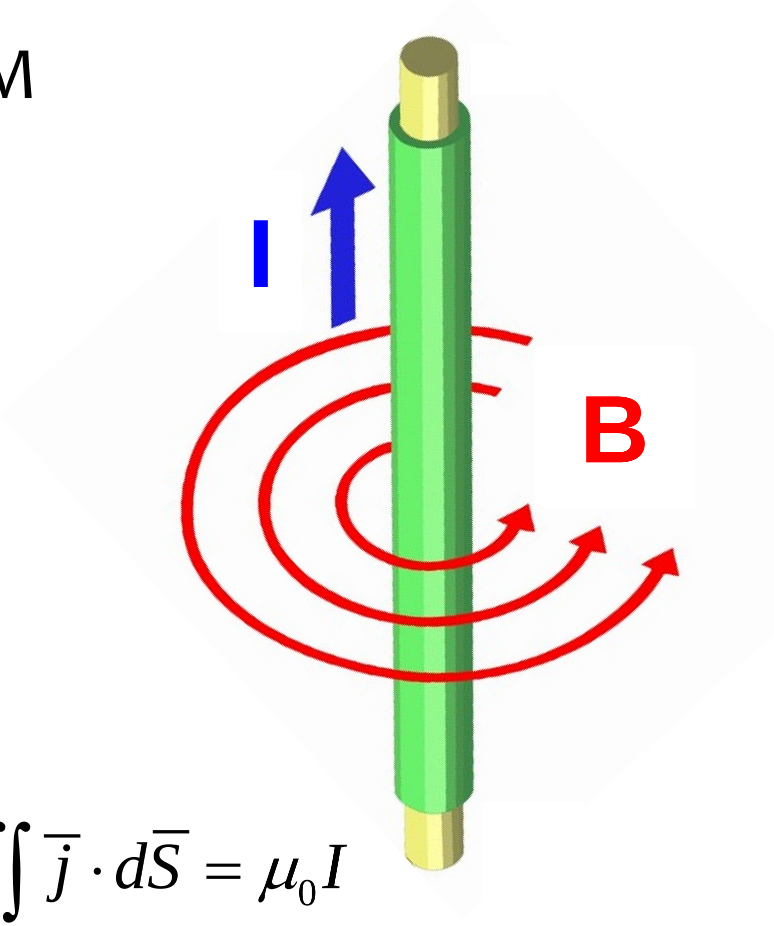
$$\shortparallel$$

$$\overbrace{\sum_i \int_{L^i} \overline{A} \cdot d\overline{r}} = \int_L \overline{A} \cdot d\overline{r}$$

$$\boxed{\oint_L \overline{A} \cdot d\overline{r} = \iint_S rot\,\overline{A} \cdot d\,\overline{S}}$$



TARGET PROBLEM



Now we can calculate the magnetic field $\overline{B}$ at a distance a from the conductor.

Ampere's law $$rot\overline{B} = \mu_0 \overline{j}$$

Where $\overline{j}$ is the current density:

$$\oint_L \overline{B} \cdot d\overline{r} = \iint_S rot\,\overline{B} \cdot d\overline{S} = \iint_S \mu_0 \overline{j} \cdot d\overline{S} = \mu_0 \iint_S \overline{j} \cdot d\overline{S} = \mu_0 I$$

Stokes

Ampere

$$I = \iint_S \overline{j} \cdot d\overline{S}$$

THE GREEN FORMULA IN THE PLANE

THEOREM *(7.1 in the textbook)*

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_L \left(Pdx + Qdy \right)}$$

PROOF

We can start from Stokes' theorem $\displaystyle \oint_L \overline{A} \cdot d\overline{r} = \iint_S rot\,\overline{A} \cdot d\,\overline{S}$

$$\oint_L \overline{A} \cdot d\overline{r} = \oint \left(A_x dx + A_y dy + A_z dz \right) = \oint \left(A_x dx + A_y dy \right)$$

But we are in a plane,
so we can assume A=(A_x,A_y,0)

$$\iint_S rot\,\overline{A} \cdot d\,\overline{S} = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \underbrace{\hat{e}_z \cdot \hat{e}_z}_{=1} dxdy$$

$$rot\,\overline{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & 0 \end{vmatrix}$$

$$\iint_D \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy = \oint_L \left(A_x dx + A_y dy \right)$$

which is the Green formula
for P=A_x and Q=A_y

CURL FREE FIELD AND SCALAR POTENTIAL

DEFINITION: A vector field $\overline{A}$ is "curl free" if $rot\ \overline{A} = 0$ *Sometimes called "irrotational"*

THEOREM *(7.5 in the textbook)* $\boxed{rot\ \overline{A} = 0 \Leftrightarrow \text{has a scalar potential } \phi,\ \overline{A} = grad\,\phi}$

PROOF

(1) $rot\ \overline{A} = 0$

$$\oint_L \overline{A}\cdot d\overline{r} = \iint_S rot\,\overline{A}\cdot d\,\overline{S} = 0$$

If the circulation is zero, then the field is conservative and has a scalar potential. *See theorem 4.5 in the textbook.*

(2) $\overline{A} = grad\,\phi$

$$rot\ \overline{A} = rot\ grad\,\phi = rot\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y}, \ldots, \ldots\right) = (0,0,0)$$

SOLENOIDAL FIELD AND VECTOR POTENTIAL

DEFINITION: A vector field $\overline{B}$ is called **solenoidal** if $div\overline{B} = 0$

DEFINITION: The vector field $\overline{B}$ has a vector potential $\overline{A}$ if , $\overline{B} = rot\overline{A}$

THEOREM *(7.7 in the textbook)* $\boxed{\overline{B} \text{ has a vector potential } \overline{A}, \quad \overline{B} = rot\overline{A} \Leftrightarrow div\overline{B} = 0}$

PROOF

(1) $\overline{B}$ has a vector potential $\Rightarrow \overline{B} = rot\overline{A} \Rightarrow div\overline{B} = div\left(rot\overline{A}\right) = 0$

(2) $div\overline{B} = 0$

Let's try to find a solution $\overline{A}$ to the equation $\overline{B} = rot\overline{A}$

We start looking for a particular solution A* of this kind:

$$\overline{A}^* = \left(A_x^*(x, y, z), A_y^*(x, y, z), 0\right)$$

CURL FREE FIELD AND SCALAR POTENTIAL

PROOF

Assuming $\bar{B}=rot\bar{A}$ we obtain:

$$-\frac{\partial A_y^*}{\partial z} = B_x \qquad \Rightarrow \qquad A_y^*(x,y,z) = -\int_{z_0}^{z} B_x(x,y,z)dz + F(x,y)$$

$$\frac{\partial A_x^*}{\partial z} = B_y \qquad \Rightarrow \qquad A_x^*(x,y,z) = \int_{z_0}^{z} B_y(x,y,z)dz + G(x,y)$$

$$\frac{\partial A_y^*}{\partial x} - \frac{\partial A_x^*}{\partial y} = B_z \qquad \Rightarrow \qquad -\int_{z_0}^{z} \frac{\partial B_x}{\partial x}dz + \frac{\partial F}{\partial x} - \int_{z_0}^{z} \frac{\partial B_y}{\partial y}dz - \frac{\partial G}{\partial y} = B_z$$

$$\downarrow$$

But $div\bar{B}=0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = -\frac{\partial B_z}{\partial z} \longrightarrow \underbrace{\int_{z_0}^{z} \frac{\partial B_z}{\partial z}dz}_{= B_z(x,y,z) - B_z(x,y,z_0)} + \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} = B_z \Rightarrow \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} = B_z(x,y,z_0)$

A solution to this equation is: $\begin{cases} F(x,y) = 0 \\ G(x,y) = -\int_{y_0}^{y} B_z(x,y,z_0)dy \end{cases}$

$$\bar{A}^* = \left(\int_{z_0}^{z} B_y(x,y,z)dz - \int_{y_0}^{y} B_z(x,y,z_0)dy, \quad -\int_{z_0}^{z} B_x(x,y,z)dz, \quad 0 \right)$$

The general solution can be found using $\bar{B}=rot\bar{A}$:

$$rot\left(\bar{A} - \bar{A}^*\right) = \bar{B} - \bar{B} = 0 \qquad \Rightarrow \qquad \bar{A} - \bar{A}^* = grad\psi \qquad \Rightarrow \qquad \bar{A} = \bar{A}^* + grad\psi$$

WHICH STATEMENT IS WRONG?

1- The curl of a vector field is a scalar (yellow)

2- The curl is related to the line integral of a field along a closed surface (red)

3- Stokes' theorem translates a line integral into a surface integral (green)

4- The Stokes' theorem can be applied only to a closed curve (blue)